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THESIS

AN IMPROVEMENT OF SOUTHWELL'S METHOD
FOR
DETERMINING BUCKLING LOADS

by

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June 1972

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ABSTRACT

If an elastic bar with some initial imperfection is subjected to increasing compressive axial load, the lateral deflection, measured anywhere along its length, increases monotonically; the primary buckling mode becoming more and more dominant in the deflected shape. A classical approximate technique, due to R. V. Southwell, correlates axial load and lateral deflection for determination of the primary buckling load of the bar. However, Southwell's approximation may be inaccurate if the primary mode does not predominate in the deflected shape. The present thesis proposes a technique which employs axial loading, but also transverse loading to insure dominance of the primary mode in the deflected shape.

The technique allows experimental determination of the second buckling load, again using appropriate transverse loads to insure dominance of that mode in the deflected shape.

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NOMENCLATURE

a_n	Coefficients which characterize deflection due to loads
b_n	Coefficients which characterize initial deflection
e_i	Error associated with the ith data point
h	Height of column cross-section (in y direction)
i,j,n	Integer subscripts
k	Lateral stiffness of column (lateral load to produce unit lateral deflection)
v	Deflection divided by corresponding load (δ/P)
w	Width of column cross-section
x	Axis parallel to applied axial load
y	Axis parallel to applied lateral load
B	Flexural rigidity EI
C_n	Compact notation for specified function of T and P_n
E	Young's modulus
I_n, J_n, K_n	Constants arising from evaluation of certain specified integrals
L	Effective length of column
P	Compressive axial load
P_n	Buckling load for the nth mode
Q	Lateral load
S	Sum of the squares of e_i
T	Tensile axial load
U	Strain energy
V	Voltage
W	Work
Y_n	Orthogonal functions

δ	Measured deflection
ρ	$1/k$

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I. INTRODUCTION

The problem of determining the elastic buckling load of a column has been one of both theoretical interest and practical importance. Euler's theory for initially straight columns has been accepted and widely used for many years. R. V. Southwell [5] treated the problem of initially imperfect columns from both theoretical and experimental viewpoints. If, before loading, the column has some small distortion, usually a superposition of many modes, the application of a compressive axial load P magnifies the amplitude of the nth mode by a factor $P_n/(P_n-P)$, where P_n is the nth mode buckling load. As P increases, magnification of the higher mode imperfections ($n > 1$) becomes relatively insignificant as compared to that of the first mode. If the column is prismatic and the deflection is measured at mid-span, even numbered modes make no contribution at all, and the dominance of the first mode is even more pronounced. Southwell postulated that to a close approximation $\delta = \delta_1/(1-P/P_1)$, where δ_1 is the first mode imperfection and δ is the mid-span deflection. Alternatively $P_1 v - \delta = \delta_1$ where $v = \delta/P$. Accordingly, if a plot of v vs. δ is made, a straight line with inverse slope equal to P_1 results, provided P is large enough to assure first mode dominance.

Successful use of Southwell's method for determining buckling loads depends upon the unloaded column having an initial lateral deflection in which there is a significant component of the first buckling mode. If this is not the case, the desired linearity of (an appropriate) plot will not be observed and it will not be possible to make an accurate determination of the first buckling

load. This paper presents an extension of Southwell's method. The proposed method does not rely on accidental imperfection. Lateral loads are used to insure dominance of the nth mode in the deflected shape when determining the nth buckling load. By using appropriate lateral loads, the buckling load for any mode may be determined, although only the first two are considered here.

II. THEORETICAL ANALYSIS

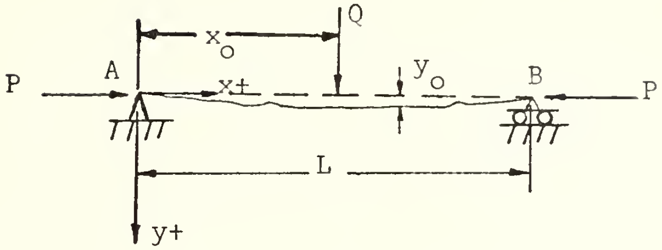

Figure 2.1
Loading Diagram

A. GENERAL CASE

Figure 2.1 represents a bar, with either variable or constant cross-section, having some initial imperfection and subjected to axial loading P and lateral loading Q. All forces and axes are shown in their positive sense. The solution for the deflected shape may be found using a linear combination of orthogonal functions to represent the initial deflection and the deflection under load. The case represented here is for a single transverse load, but the method may be extended for any number of loads. Here and elsewhere it is assumed that all loads and deflections are confined to a single plane, namely the x-y plane.

Let y_o be the initial ordinate, measured from the chord AB and let y_1 be the deflection due to external loads. The total ordinate after the loads have been applied is $y = y_o + y_1$.

Assume $y_o = b_1Y_1 + b_2Y_2 + b_3Y_3 + \dots$ (2.1)

$$y_1 = a_1Y_1 + a_2Y_2 + a_3Y_3 + \dots \quad (2.2)$$

where $Y_n = Y_n(x)$ $(n = 1,2,3,\ldots)$ is a set of orthogonal functions in the interval (0,L) which satisfy the differential equation B-1a subject to conditions B-1b (see Appendix B). As the loads are applied, the distance between A and B decreases by an amount

$$\lambda - \lambda_o = \frac{1}{2}\int_0^L \left[\frac{d(y_1+y_o)}{dx}\right]^2 dx - \frac{1}{2}\int_0^L \left[\frac{dy_o}{dx}\right]^2 dx$$

$$= \frac{1}{2}\int_0^L \left[(y')^2 - (y_o')^2\right] dx \qquad (2.3)$$

where $y' = \Sigma(a_i+b_i)\,Y_i'$; $y_1' = \Sigma a_i Y_i'$; $y_o' = \Sigma b_i Y_i'$

and the prime denotes differentiation with respect to x.

$$(y')^2 = \Sigma\Sigma(a_i+b_i)(a_j+b_j)Y_i'\,Y_j';\ (y_o')^2 = \Sigma\Sigma b_i\,b_j\,Y_i'\,Y_j'$$

$$\therefore\ \lambda - \lambda_o = \frac{1}{2}\,\Sigma\left\{\left[(a_i+b_i)^2 - b_i^2\right]\int_0^L (Y_i')^2 dx\right\}$$

$$= \Sigma\left(\frac{a_i^2}{2} + a_i b_i\right)\int_0^L (Y_i')\,dx$$

$$= \Sigma\left(\frac{a_i^2}{2} + a_i b_i\right) J_n \qquad (2.4)$$

For definitions and properties of the functions appearing here and below, see Appendix B.

Now consider the energy variations due to some small additional deflection $da_n Y_n$ and assume da_n is positive in the same sense as Q.

The work done by the force P is

$$dW_P = P\,\frac{\partial(\lambda - \lambda_o)}{\partial a_n}\cdot da_n = P(a_n+b_n)\cdot J_n\cdot da_n \qquad (2.5)$$

The work done by the force Q (located at $x = x_0$) is

$$dW_Q = QY_n(x_0)\cdot da_n \qquad (2.6)$$

The increase in strain energy is

$$dU = \frac{\partial U}{\partial a_n} \cdot da_n = \frac{\partial}{\partial a_n} \left[\frac{1}{2} \int_0^L EI(x) \, (y_1'')^2 dx \right] da_n$$

$$= \frac{\partial}{\partial a_n} \left[\frac{1}{2} \int_0^L B(y_1'')^2 dx \right] da_n$$

where $B = EI(x) = B(x)$

$$dU = \frac{\partial}{\partial a_n} \frac{1}{2} \Sigma\Sigma \left[a_i \, a_j \int_0^L B \; Y_i'' \, Y_j'' \, dx \right] da_n$$

$$= \frac{\partial}{\partial a_n} \frac{1}{2} \Sigma \, a_i^2 K_i \cdot da_n$$

$$= a_n \; K_n \; da_n \qquad (2.7)$$

Equating the increase in strain energy and the work done by the external forces, an equation for the coefficients a_n may be found

$$P(a_n + b_n) J_n \cdot da_n + Q Y_n(x_0) \cdot da_n = a_n \; K_n \; da_n$$

$$a_n = \frac{Q Y_n(x_0) + P b_n \; J_n}{K_n - P J_n}$$

or since $K_n / J_n = P_n$

$$a_n = \frac{\frac{Q}{K_n} Y_n(x_0) + \frac{P}{P_n} b_n}{1 - \frac{P}{P_n}} \qquad (2.8)$$

$$\therefore \; y = y_1 + y_o = \Sigma \frac{\frac{Q}{K_n} Y_n(x_0) + b_n}{1 - \frac{P}{P_n}} \cdot Y_n(x) \qquad (2.9)$$

These terms may be ordered so that $P_1 < P_2 < P_3 \cdots$

It is obvious, for values of P approaching P_1, that if Q is sufficiently large, the first term of the series is strongly dominant. Southwell, rather than applying a lateral load Q, depended on some accidental

imperfection such that $b_1 \neq 0$. However, if the nature of the initial imperfection is such that $b_n \gg b_1$ for some $n \gg 1$, the first term does not dominate unless P is very nearly equal to P_1. Consequently, the linear relation which Southwell's analysis depends on may not appear in plots of experimental data.

Equation (2.9) also demonstrates something which does not seem to have been treated in previous literature. If the value of the applied lateral load Q is sufficient, the first term will dominate even if b_1 is not the largest b_n and even if P does not approach P_1. Furthermore, if the lateral load is applied near the column midlength, then $Y_1(x_0)/K_1 \gg Y_2(x_0)/K_2$. Thus, in effect, the first mode imperfection may be increased by the use of lateral loading, insuring dominance of the first mode in the deflected shape, regardless of the value of $P(P \leq P_1)$. By increasing the effective first mode imperfection to $b_1^* = b_1 + (QY_1(x_0)/K_1)$, the accuracy and reliability of Southwell's method can be greatly enhanced.

Another experimental procedure, which involves observing deflection as a function of lateral load Q for a few particular values of P, will be described subsequently.

For the special case of columns which are symmetric with respect to a normal plane through the center, if the lateral load is applied at mid-span, non-zero values of the coefficients a_n occur only for $n = 1,3,5, \ldots$ These coefficients attenuate rapidly $(1 \gg a_3/a_1 \gg a_5/a_1 \ldots)$ and it is evident that the first term of equation (2.9) dominates the others even if $1-P/P_1$ is not small.

This suggests that the value of P_1 might be accurately approximated even if tensile loadings (i.e., negative values of P) are used. First

term dominance is diminished slightly, but not significantly, for tensile loading. In the experimental program to be described, tensile loading was used partly for convenience, but largely to demonstrate what appears not to have been done before, namely to evaluate a compressive critical load without ever applying compression.

Assuring that the applied axial compressive force P does not closely approach the first buckling load P_1 permits the experimental determination of the second critical buckling load! To do this it is necessary to assure dominance of the second term of equation (2.9). This can be done, again using lateral loads to "excite" mainly the second mode. Generally, it is difficult to develop an experimental technique which will permit accurate establishment of the true first mode shape $Y_1(x)$ and also one which is orthogonal to it. However, if the column is symmetric with respect to a central normal plane, clearly the mode shapes are alternately symmetric and antisymmetric. Thus, antisymmetrical loading can "excite" only the second and higher antisymmetrical modes. If the lateral loading consists of two oppositely directed forces near the quarter points, fourth mode excitation will be small and the second should dominate.

B. PRISMATIC BAR

Although there is no necessity for determining the buckling load of an elastic, prismatic bar by experimental means since the theory provides complete information for that case, it is still reasonable to select such a bar for experimental confirmation of the general analysis. The particular form of the preceding results for the case of a prismatic bar may be easily found. The orthogonal functions are $Y_n = \sin n\pi x/L$. The constants K_n and J_n are then

$$K_n = \frac{EIn^4\pi^4}{2L^3} \tag{2.10}$$

$$J_n = \frac{n^2\pi^2}{2L} \tag{2.11}$$

Equation (2.9) is then

$$y = y_1 + y_0 = \Sigma\left[\left(\frac{\frac{2QL^3}{EIn^4\pi^4}\sin\frac{n\pi x_0}{L} + b_n}{1 - \frac{P}{P_n}}\right)\sin\frac{n\pi x}{L}\right] \tag{2.12}$$

For first mode determination, the load Q could be applied anywhere in (0,L) and the deflection could be measured at the same or any other position in (0,L). However, it is convenient to do both at the center, $x = L/2$. If this is done, then $x = x_o = L/2$ and equation (2.12) becomes

$$y = \frac{2QL^3}{EI\pi^4}\left(\frac{1}{1 - \frac{P}{P_1}} + \frac{1}{81(1 - \frac{P}{9P_1})} + \frac{1}{625(1 - \frac{P}{25P_1})} + \cdots\right)$$

$$+\left(\frac{b_1}{1 - \frac{P}{P_1}} - \frac{b_3}{1 - \frac{P}{9P_1}} + \cdots\right) \tag{2.13}$$

If instead of a compressive load P, there is a tensile load T, clearly $T = -P$ and equation (2.13) becomes

$$y = \frac{Q}{k}\left(\frac{1}{1 + \frac{T}{P_1}} + \frac{1}{81(1 + \frac{T}{9P_1})} + \cdots\right) + y^* \tag{2.14}$$

where the quantity k represents lateral stiffness

$$k = EI\pi^4/2L^3$$

and

$$y^* = \left(\frac{b_1}{1 + \frac{T}{P_1}} - \frac{b_3}{1 + \frac{T}{9P_1}} + \frac{b_5}{1 + \frac{T}{25P_1}} - \cdots\right)$$

With the application of Q at mid-span, the bar will deflect essentially in a first mode configuration, higher mode contributions being small.

C. ESTABLISHMENT OF REFERENCE LEVEL

The last term in equation (2.14) presents a problem in that its exact value is unknown and cannot be determined in the laboratory. The problem may be circumvented however. Referring to Fig. (2.2), three load cases are shown:

(a) T = Q = 0;

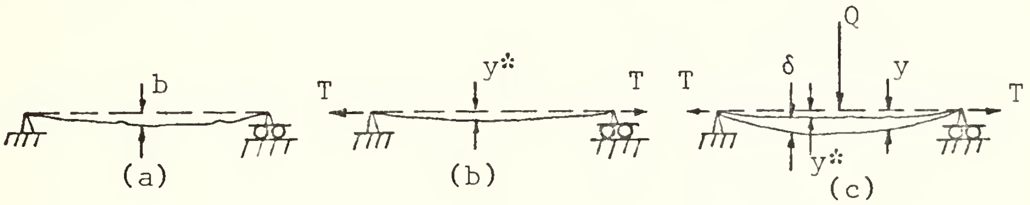

Figure (2.2)
Experimental Loading Cases

(b) Q = 0, T = T; and (c) Q = Q, T = T. With no loads applied there is some initial imperfection b. When load T is applied (Q = 0), the amplitude of the imperfection is reduced to an amount y* which is exactly equal to the last term in equation (2.14). With the load T applied and maintained, a reference level may be established for all subsequent deflections due to lateral loading. Now as the lateral load Q is applied (holding T constant), the deflection measured in the laboratory is simply

$$\delta = \frac{Q}{k}\left(\frac{1}{1 + \frac{T}{P_1}} + \frac{1}{81(1 + \frac{T}{9P_1})} + \frac{1}{625(1 + \frac{T}{25P_1})} + \cdots \right) \qquad (2.15)$$

D. NON-PRISMATIC BARS

For non-prismatic bars, equation (2.9) describes the deflected shape in terms of the applied loads, the initial imperfection, and some possibly unknown orthogonal functions Y_n. The forms of Y_1, Y_2, ... are clearly not harmonic in x, but it should still be possible to excite mainly the first characteristic mode Y_1 using a lateral load Q applied near the column midlength. The deflected shape will not be a half-sine curve as for prismatic bars, but it still will not have any intermediate nodes.

If the reference level for laboratory measurements is established as before and the measurements are taken at the point of load application $x = x_0$, the measured deflection is then

$$\delta = \frac{QY_1^2(x_0)}{K_1}\left(\frac{1}{1+\frac{T}{P_1}} + \frac{Y_2^2(x_0)K_1}{Y_1^2(x_0)K_2}\cdot\frac{1}{1+\frac{T}{P_2}} + \frac{Y_3^2(x_0)K_1}{Y_1^2(x_0)K_3}\cdot\frac{1}{1+\frac{T}{P_3}} + \ldots\right) \tag{2.16}$$

E. ANALYSIS OF EXPERIMENTAL DATA

Continuing with the general case of a non-prismatic bar, once experimental data has been obtained, it may be analyzed, using equation (2.16), by several methods. Two specific methods are outlined here.

Considering the relative magnitudes of the series terms in equation (2.16), the first is clearly the dominant term for any given value of T. If only the first term is retained as an approximation, δ is a linear function of Q for any given T since K_1, $Y_1(x_0)$ and P_1 are all constant. A plot of δ vs. Q, for a given T yields a straight line with an inverse slope equal to

$$k_P = \frac{K_1}{Y_1^2(x_0)}\left(1+\frac{T}{P_1}\right) = k\left(1+\frac{T}{P_1}\right) \tag{2.17}$$

Thus k_p is a linear function of T, indicating that a plot of k_p vs. T will have a slope equal to k/P_1 and an intercept equal to k. By simply dividing the intercept by the slope a value of P_1 may be found.

A second method, which may offer greater accuracy, is a least squares approach using P_1 as a parameter. Consider again equation (2.16). For beams which are "nearly prismatic" the functions $Y_n(x)$ will be "nearly" sine functions so that an estimate of the relative values of the first several coefficients $Y_n^{\,2}(x_0)K_1/Y_1^{\,2}(x_0)K_n$ can be made by assuming the $Y_n(x)$ functions are indeed sines. Similarly the ratios P_2/P_1, P_3/P_1, P_5/P_1 ..., can be estimated to be 4, 9, 25, ... This provides a way of estimating the contributions of terms beyond the first. If these added terms can be at least estimated, they can be retained and used advantageously; if not, only the first term is retained.

For any given value of the parameter P_1, equation (2.16) maybe written

$$\delta_i = \frac{Q_i}{k}\left(\frac{1}{1+\frac{T_i}{P_1}} + \frac{Y_2^2(x_0)K_1}{Y_1^2(x_0)K_2}\cdot\frac{1}{1+\frac{T_i}{P_2}} + \frac{Y_3^2(x_0)K_1}{Y_1^2(x_0)K_3}\cdot\frac{1}{1+\frac{T_i}{P_3}} + \ldots\right)$$

$$= \frac{Q_i}{k}\,C_i = Q_i\,\rho\,C_i \qquad (2.18)$$

where $\rho = 1/k$. In the series $C_i = C(T_i)$, only those terms are retained which can be estimated with some confidence. Let e_i be the error associated with the ith data point.

$$e_i = Q_i\,\rho\,C_i - \delta_i \qquad (2.19)$$

Let S equal the sum of the squares of e_i for n data points.

$$S = \sum_{i=1}^{n} e_i^2 \qquad (2.20)$$

To find the most probable (in the sense of least squares) value of ρ and P_1 from the experimental data, proceed as follows:

$$\frac{1}{2}\frac{\partial S}{\partial \rho} = 0 = \sum_{i=1}^{n} Q_i C_i e_i \tag{2.21}$$

$$\sum_{i=1}^{n} (Q_i C_i)^2 \rho - \sum_{i=1}^{n} \delta_i Q_i C_i = 0 \tag{2.22}$$

$$\rho = \frac{\sum_{i=1}^{n} \delta_i Q_i C_i}{\sum_{i=1}^{n} (Q_i C_i)^2} \tag{2.23}$$

Now find ρ and S for different values of the parameter P_1. The quantity S is a minimum when the most probable value of P_1 is used, also yielding a most probable value of ρ.

For the case of a prismatic bar, equation (2.15) describes the deflected shape. The two methods just outlined are, of course, applicable to this simplified case. All of the terms in equation (2.15) are known and hence all may be used to improve the accuracy of the least squares analysis if desired.

Both of the methods outlined above may be easily programmed for the digital computer. Appendix A gives program listings for the prismatic bar case.

F. DETERMINATION OF THE SECOND BUCKLING LOAD

Determination of the second mode buckling load using the theory outlined above is straightforward assuming that EI(x) = EI(L-x). If an upward load -Q is applied at $x = x_0$ and a downward load Q is applied at $x = L-x_0$, the resulting equation for deflection is similar to equation (2.12). (It is convenient to apply the lateral loads at $x = L/4$ and $x = 3L/4$.) Such lateral loading does not excite odd

numbered modes, only even numbered modes, and if Q is sufficiently large, the term corresponding to the second mode can be made to dominate. The first, third, fifth, ... terms are negligible, and the fourth, sixth, ... are small compared to the second. Thus, the sort of data reduction discussed above will lead to an evaluation of the second mode.

If $EI(x) \neq EI(L-x)$, that is if there is no symmetry with respect to a normal plane at $x = L/2$, it may be quite difficult to apply a lateral loading system which does not excite the first mode. A suggested approach is to apply an upward load at $x = L/4$ and an equal downward load at a point "near" $x = 3L/4$. Unless the second point is selected correctly, the mixture of first and second modes thus excited will not provide a linear plot of data. One would have to try several such points in order to obtain the desired sort of plot which could then be interpreted so as to yield the second buckling load.

III. EXPERIMENTAL PROCEDURE

A. DESIGN

During the design and fabrication of the test apparatus, strict attention was paid to simplicity. The major problems which arose where how to best simulate pinned end conditions in the laboratory, and how to apply the required axial and lateral loads. Several designs were considered which could easily transmit an applied compressive axial load, but they could not easily support an applied lateral load without introducing unknown end moments. Application of a compressive axial load using a universal testing machine could be accomplished, but lateral loads could not be easily applied or measured. Dead weights seemed to be the simplest means of applying both axial and lateral loads. The final design of the test apparatus is shown schematically in Fig. (3.1a).

B. FABRICATION

Compressive axial loads could not be easily applied using dead weights so it was decided to use tensile loading for convenience. This also allowed determination of higher mode buckling loads. However, there is a penalty for using tensile axial loads. This will be discussed in Section IV-B. Pinned end conditions were simulated using steel cylinders on either end. The cylinders were polished and rested on a finished surface. A mixture of graphite, molybdenum sulfide, and machine oil was used as a lubricant to eliminate dry friction between these cylinders and the test bed. Flexible steel cable was used to transmit the axial load T. The cable was brazed to a steel ring for

attachment to the end cylinders. A slot was cut in each end cylinder, perpendicular to the longitudinal axis and the ring was inserted. A pin which provided a "knife edge" contact was inserted parallel to the longitudinal axis of the cylinder, passing through the eye of the ring, see Fig. (3.1b). The knife edge of the pin ran directly along the longitudinal axis of the cylinder. The effective length of the column was measured from centerline to centerline of the end cylinders and was 19.00 inches. Since a near-zero moment exists near these centers, changes in EI near the ends have negligible effect on bending, and the effective column length may be safely considered as the distance between centers of the end cylinders. The column itself was 1/4" x 1/8" cold drawn 1018 steel, with a modulus of elasticity $E = 29.4 \times 10^6$ psi. It was brazed to the end cylinders. The lateral load was applied with a knife edge and dead weights as shown in the figure. Deflection measurements were taken with a simple wire probe attached to a vertically mobile micrometer. The knife edge assembly used for application of lateral loads had a hole drilled in the flat upper surface which was filled with mercury. The probe was lowered until contact with the mercury was made, completing a simple neon light bulb circuit, see Fig. (3.2). For second mode buckling tests, two lateral loads were applied, one at each quarter point. The downward load was applied as before. The upward load was also applied using a knife edge and dead weights. A pulley was used to convert dead weight into upward force. Since an upward force was applied at one end of the beam, the cylinder on that end had to be restrained against upward motion. This was done with a simple "hood", see Fig. (3.3).

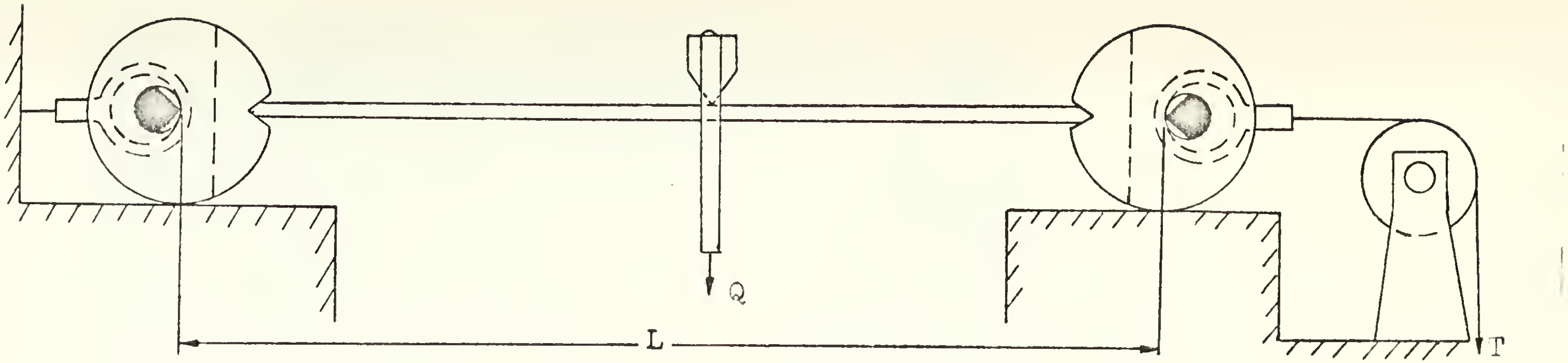

Figure 3.1a
SCHEMATIC OF TEST APPARATUS

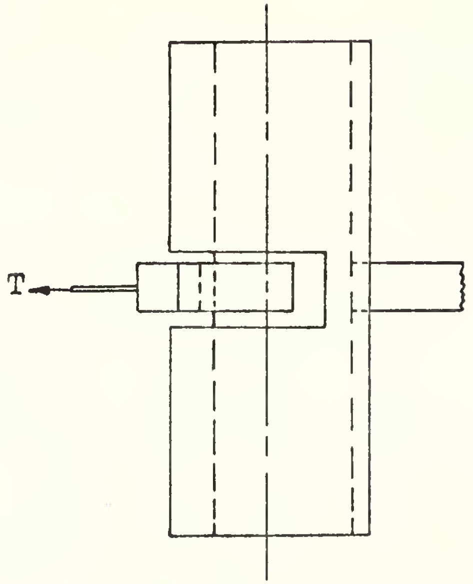

Figure 3.1b
TOP VIEW OF END CYLINDER

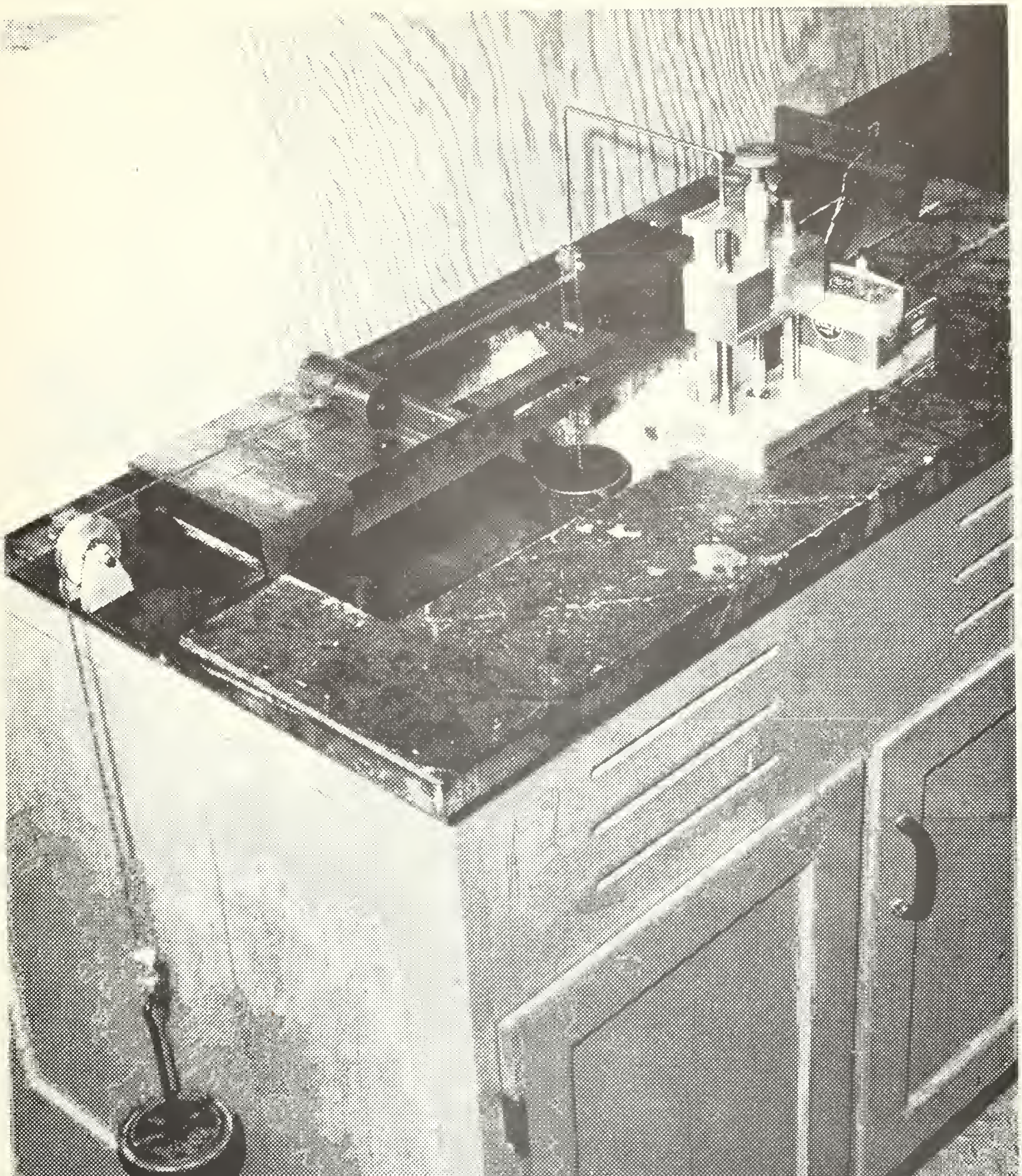

Figure 3.2
APPARATUS FOR DETERMINATION OF FIRST BUCKLING LOAD

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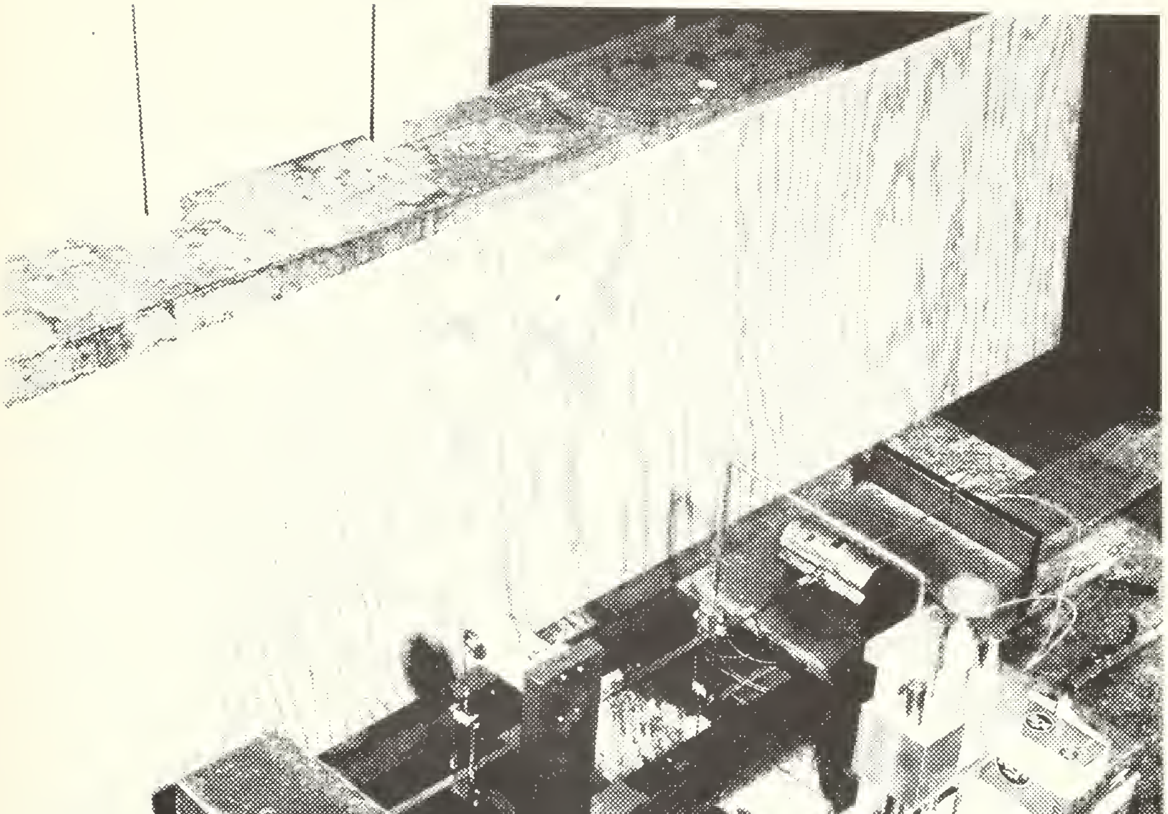

Figure 3.3
APPARATUS FOR DETERMINATION OF SECOND BUCKLING LOAD

C. TESTS

The reference level for testing was established before each test by first applying the axial load T, and then taking a reading from the micrometer with the knife edge assembly in place. The weight of the knife edge caused some small deflection, but this may be treated in the same manner as the initial imperfections were. Once the reference level was established, the lateral load was increased from 0.0 to 4.0 pounds in half-pound increments with a final 1/4 pound increment to give a maximum load of 4.25 pounds. No greater lateral load was applied so as to assure that plastic bending did not take place. This was done for 15 different values of T ranging from 0.0 to 42.0 pounds, in three-pound increments. For each such loading, the values of loads and corresponding micrometer readings were recorded.

The same method of establishing a reference was used for second mode buckling tests. An upward load (-Q) was applied at $x = L/4$ and a downward load (Q) was applied at $x = 3L/4$. The value of Q ranged from 0.0 to 6.5 pounds, in half-pound increments. The range of Q could have been extended another two pounds without causing plasticity, but the effects of friction became evident above 6.5 pounds, since increases in Q caused almost immeasurable changes in deflection. The values of T ranged from 0.0 to 41.0 pounds in five-pound increments, the odd pound arising from the weight pan used. A few anomalies in the data for second mode were noted and will be discussed subsequently. Several attempts to find the cause of the discrepancies were made, but proved fruitless.

Once the data was obtained, it was analyzed using both of the methods described previously.

As a means of substantiating earlier test results, another type of experiment was done to determine the primary buckling load, using axial compression and interpreting the data according to Southwell's method. The apparatus used for this test is shown in Fig. (3.4). A cantilever beam, embedded in a vertically mobile "wall" was used to apply a compressive axial load to the column. The cantilever was instrumented with four strain gages, two mounted parallel to the direction of principal strain, and two perpendicular to it to compensate for Poisson's effect. The gages were connected in a four-arm bridge circuit and hooked to a Baldwin Model SR-4 Strain Indicator -- Type N. The column was also instrumented with strain gages. One gage was mounted on each side of the column at mid-span, parallel to the direction of principal strain. These two gages were designed specifically for use on mild steels and were self-compensating for small ambient temperature changes. These two gages were connected in a two-external-arm bridge hooked to an Ellis Bridge Amplifier and Meter (BAM-1). The output of the BAM was monitored on a digital voltmeter.

The cantilever strain gages were calibrated using dead weight tip loads, providing a direct correlation between indicated strain and applied load.

Before each test, the no-load reference for the cantilever bridge was noted. A small load was then applied to hold the column vertical, and the reference reading for the column bridge was noted. Preliminary tests indicated that the Southwell plot became linear for $P \geq .5P_1$. Therefore, the axial load, including the weight of the end cylinder,

Once the data was [illegible]

methods described previously [illegible]

at a number of [illegible]

of acquisition [illegible]

well documented [illegible]

method. The [illegible]

additional [illegible]

with a [illegible]

[illegible]

[illegible]

with the [illegible]

bridge [illegible]

Type [illegible]

can [illegible]

direction [illegible]

really [illegible]

method [illegible]

[illegible]

[illegible]

The [illegible]

loads, yielding [illegible]

applied load.

[illegible]

was [illegible] a small [illegible]

and the [illegible]

loads indicated [illegible]

Thereafter the [illegible]

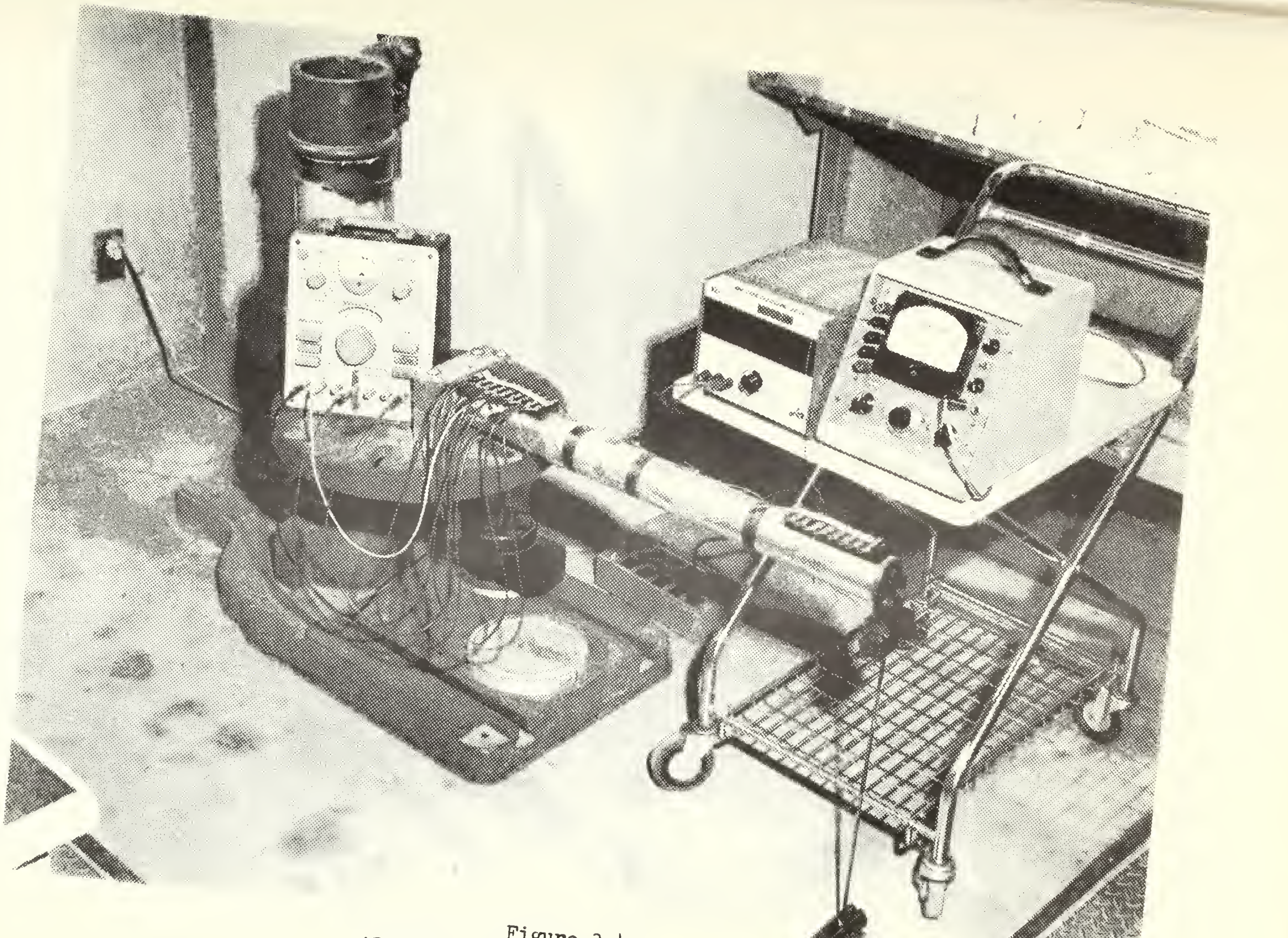

Figure 3.4
APPARATUS USED FOR SOUTHWELL TEST

was increased from 16. to 26. pounds in one-pound increments. At each load increment the strain readings from the cantilever bridge and the voltage readings from the column bridge were recorded, the strain reading being proportional to applied axial load, and the voltage change ($\Delta V = V - V_o$) being proportional to column deflection.

$\Delta V/P$ versus ΔV was then plotted for each of four runs and the value of P_1 was determined by the reciprocal of the slope. A computer program was utilized to obtain the best fitted line, for each run, again using a linear least squares fit, and the average value of P_1 for all runs was calculated (see Appendix A).

A standard tensile test was run on a specimen of the material used for the column to determine Young's modulus experimentally. The result was $E = (29.4 \pm .1) \times 10^6$ psi.

IV. RESULTS AND DISCUSSION

A. RESULTS

The figures tabulated in Table I represent the experimentally determined values of the buckling load for both first and second modes. As described earlier (section 2-E) two methods were used to analyze the data: one being a linear approximation, hereafter method I; the other being a least squares analysis retaining any desired number of terms in equation (2.15), hereafter method II. The average value of P_1 arrived at by use of Southwell's method is also tabulated.

	Method I	Method II		Southwell's Method
Number of terms retained	1	1	4	——
1st mode buckling load (lbs.)	31.0	32.0	31.2	32.1*
2nd mode buckling load (lbs.)	127.8	110.5	109.0	——

Table I

Summary of Results

Typical plots of δ vs. Q and k_p vs. T for method I are shown in Fig. (4.1) and (4.2) respectively. A typical plot for determination of P_1 by Southwell's method is shown in Fig. (4.3). Compiled data for all tests is listed in Tables II - V.

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* Average value of four tests.

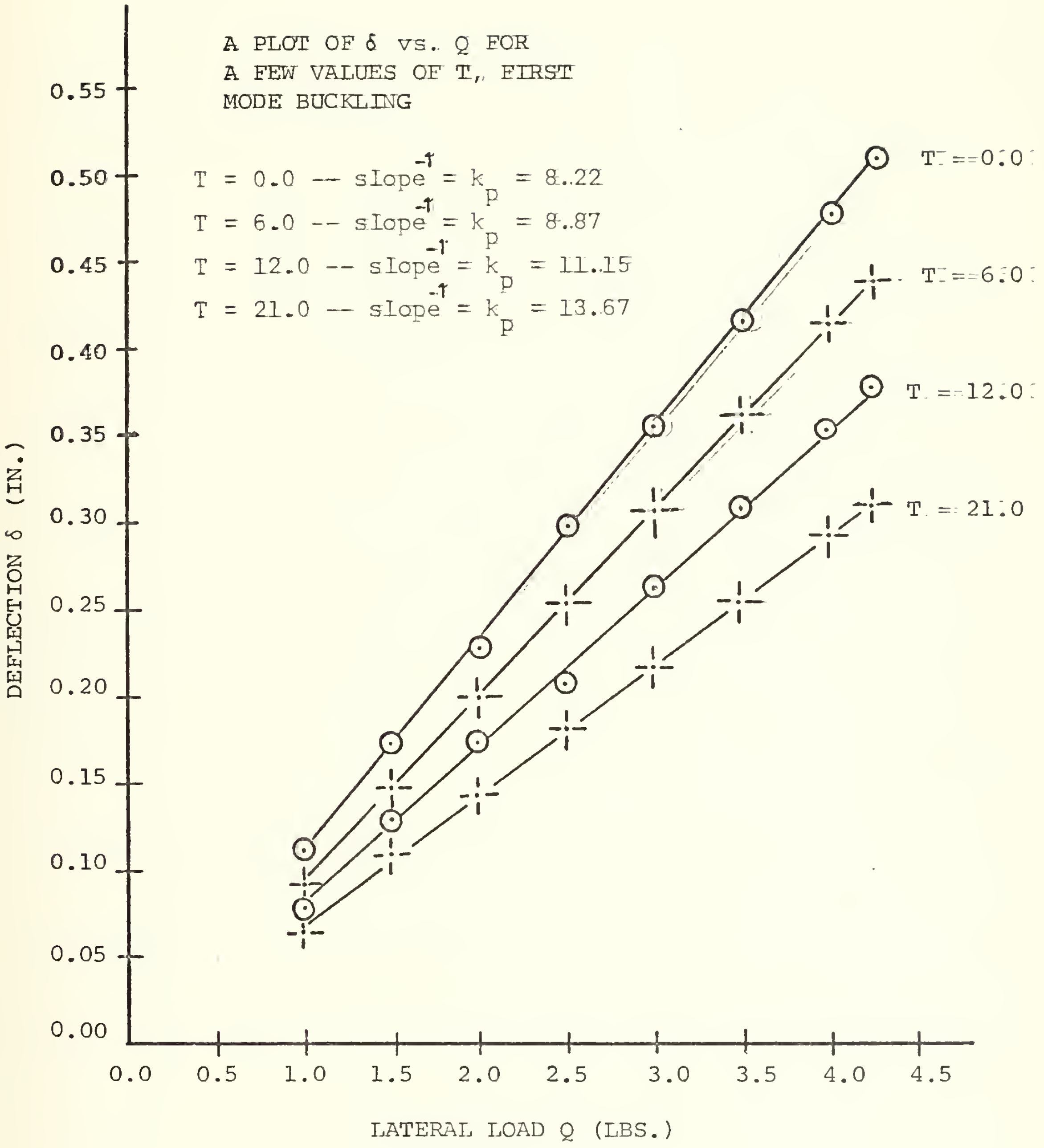

A PLOT OF δ vs. Q FOR
A FEW VALUES OF T, FIRST
MODE BUCKLING
T = 0.0 -- $\text{slope}^{-1} = k_p$ = 8.22
T = 6.0 -- $\text{slope}^{-1} = k_p$ = 8.87
T = 12.0 -- $\text{slope}^{-1} = k_p$ = 11.15
T = 21.0 -- $\text{slope}^{-1} = k_p$ = 13.67
T = 0.0
T = 6.0
T = 12.0
T = 21.0
DEFLECTION δ (IN.)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
LATERAL LOAD Q (LBS.)

Fig. (4.1)

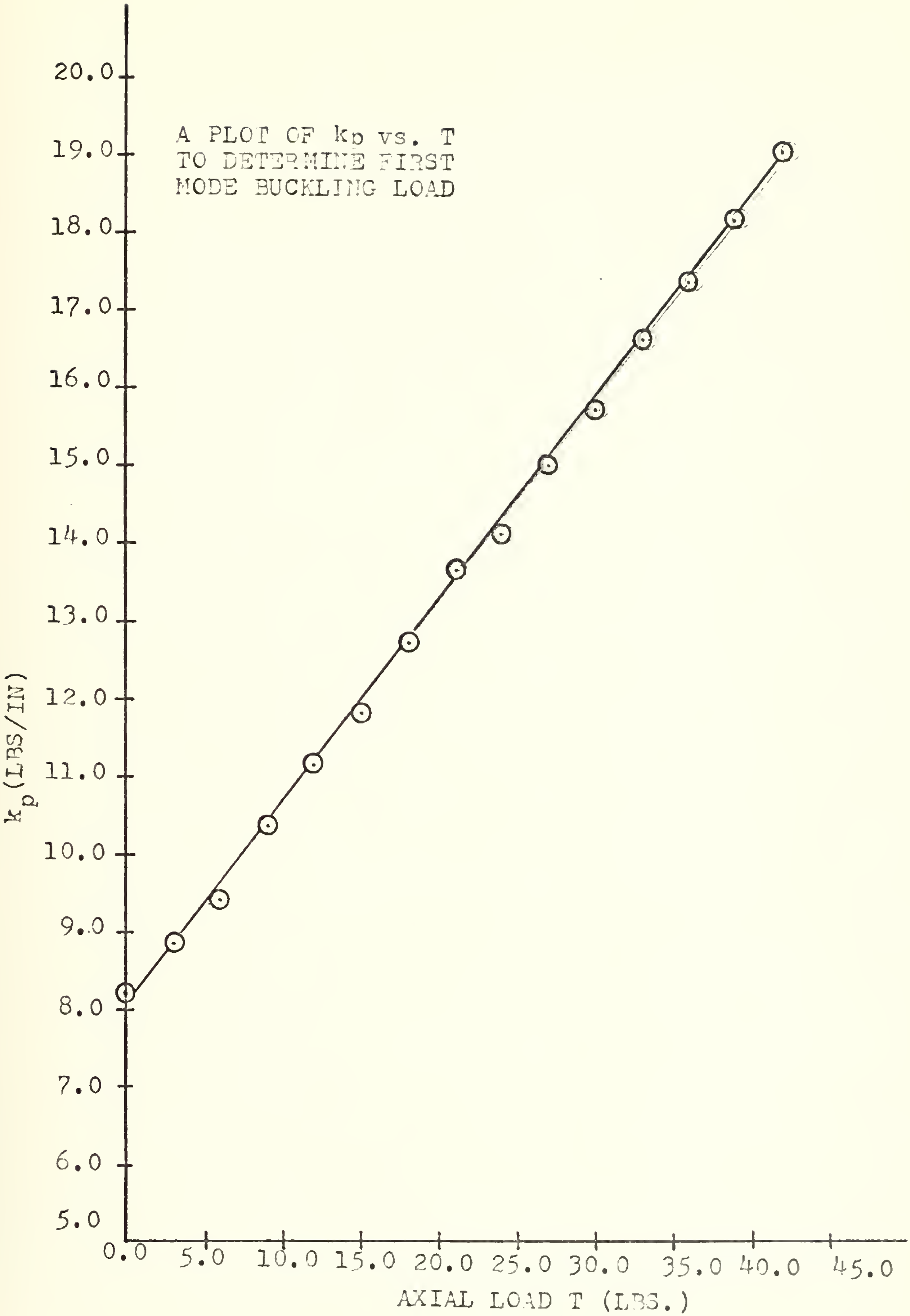
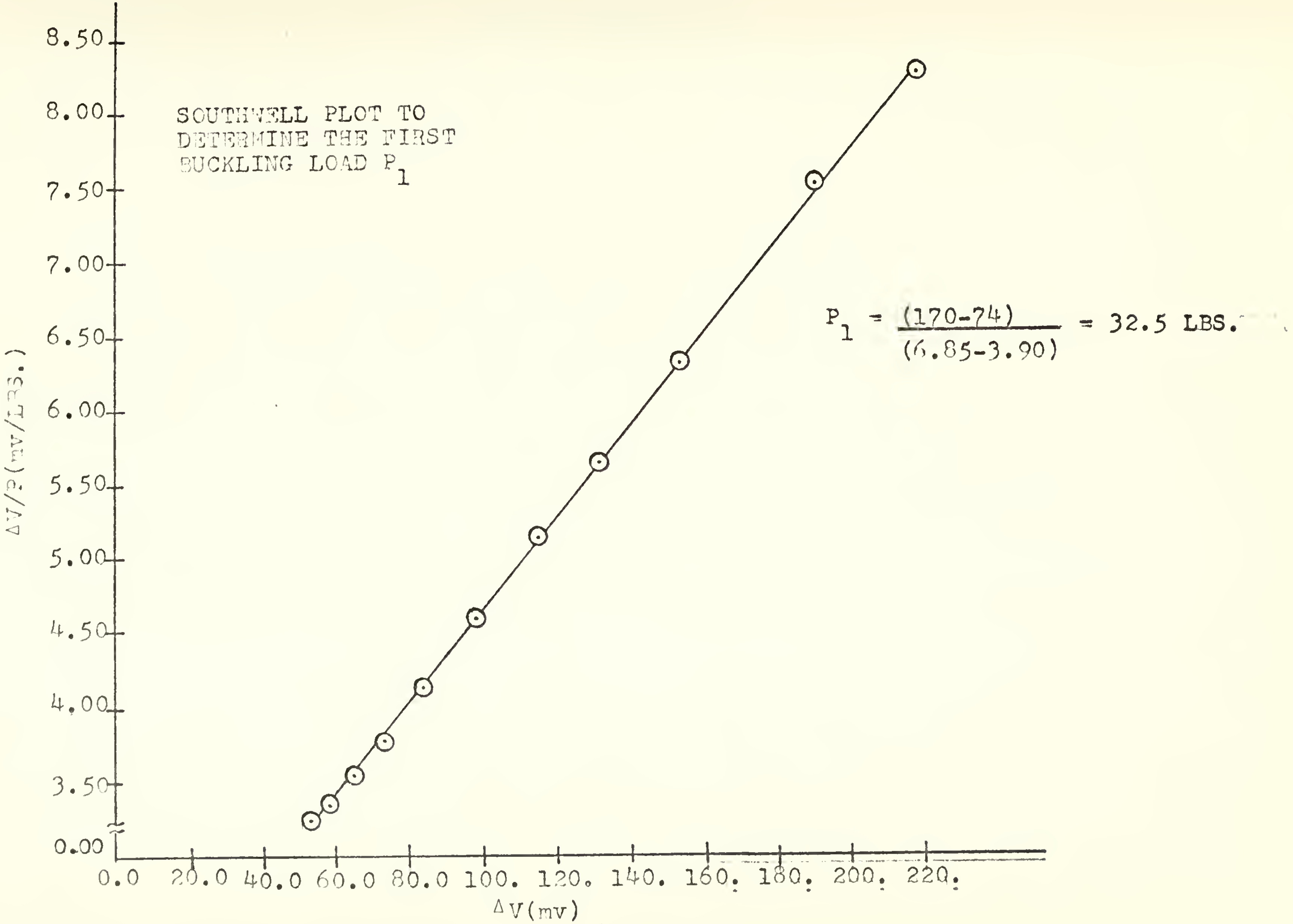

A PLOT OF kp vs. T
TO DETERMINE FIRST
MODE BUCKLING LOAD
20.0
19.0
18.0
17.0
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
kp (LBS/IN)
0.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0
40.0
45.0
AXIAL LOAD T (LBS.)

FIG. (4.2)



SOUTHWELL PLOT TO DETERMINE THE FIRST BUCKLING LOAD P_1

$$P_1 = \frac{(170-74)}{(6.85-3.90)} = 32.5 \text{ LBS.}$$

Fig. (4.3)

1ST MODE BUCKLING DATA

T	Q	M.R.	δ	T	Q	M.R.	δ	T	Q	M.R.	δ	T	Q	M.R.	δ
0.0	0.0	.6730	ref.	3.0	0.0	.6680	ref.	6.0	0.0	.6720	ref.	9.0	0.0	.6660	ref.
0.0	1.0	.7840	.1110	3.0	1.0	.7763	.1083	6.0	1.0	.7640	.0920	9.0	0.0	.7467	.0807
0.0	1.5	.8468	.1738	3.0	1.5	.8236	.1556	6.0	1.5	.8171	.1451	9.0	1.5	.7943	.1283
0.0	2.0	.9062	.2332	3.0	2.0	.8845	.2165	6.0	2.0	.8703	.1983	9.0	2.0	.8412	.1752
0.0	2.5	.9712	.2982	3.0	2.5	.9430	.2750	6.0	2.5	.9229	.2509	9.0	2.5	.8920	.2260
0.0	3.0	1.0305	.3575	3.0	3.0	.9935	.3255	6.0	3.0	.9773	.3053	9.0	3.0	.9375	.2715
0.0	3.5	1.0882	.4152	3.0	3.5	1.0580	.3900	6.0	3.5	1.0320	.3600	9.0	3.5	.9860	.3200
0.0	4.0	1.1498	.4768	3.0	4.0	1.1150	.4470	6.0	4.0	1.0832	.4112	9.0	4.0	1.0350	.3690
0.0	4.25	1.1805	.5075	3.0	4.25	1.1342	.4662	6.0	4.25	1.073	.4353	9.0	4.25	1.0592	.3932
12.0	0.0	.6568	ref.	15.0	0.0	.6532	ref.	18.0	0.0	.6560	ref.	21.0	0.0	.6565	ref.
12.0	1.0	.7388	.0820	15.0	1.0	.7338	.0806	18.0	1.0	.7275	.0715	21.0	0.0	.7245	.0680
12.0	1.5	.7878	.1310	15.0	1.5	.7746	.1214	18.0	1.5	.7658	.1098	21.0	1.5	.7675	.1110
12.0	2.0	.8300	.1732	15.0	2.0	.8162	.1630	18.0	2.0	.8053	.1493	21.0	2.0	.8008	.1443
12.0	2.5	.8738	.2170	15.0	2.5	.8590	.2058	18.0	2.5	.8452	.1892	21.0	2.5	.8388	.1823
12.0	3.0	.9180	.2612	15.0	3.0	.9015	.2483	18.0	3.0	.8843	.2283	21.0	3.0	.8742	.2177
12.0	3.5	.9640	.3072	15.0	3.5	.9465	.2933	18.0	3.5	.9240	.2680	21.0	3.5	.9102	.2537
12.0	4.0	1.0092	.3524	15.0	4.0	.9871	.3339	18.0	4.0	.9625	.3065	21.0	4.0	.9467	.2902
12.0	4.25	1.0330	.3762	15.0	4.25	1.0089	.3557	18.0	4.25	.9822	.3262	21.0	4.25	.9650	.3085

Note: T = Applied axial load; Q = Applied lateral load; M.R. = Micrometer reading (in.)
δ = Measured deflection (in.)
All loads in lbs.

Table II

T	Q	M.R.	δ	T	Q	M.R.	δ	T	Q	M.R.	δ	T	Q	M.R.	δ
24.0	0.0	.6555	ref.	27.0	0.0	.6510	ref.	30.0	0.0	.6523	ref.	33.0	0.0	.6515	ref.
24.0	1.0	.7153	.0598	27.0	0.0	.7112	.0602	30.0	1.0	.7090	.0567	33.0	1.0	.7070	.0555
24.0	1.5	.7525	.0970	27.0	1.5	.7456	.0946	30.0	1.5	.7422	.0899	33.0	1.5	.7365	.0850
24.0	2.0	.7850	.1295	27.0	2.0	.7789	.1279	30.0	2.0	.7735	.1212	33.0	2.0	.7662	.1147
24.0	2.5	.8212	.1657	27.0	2.5	.8138	.1628	30.0	2.5	.8073	.155	33.0	2.5	.7975	.1460
24.0	3.0	.8570	.2015	27.0	3.0	.8450	.1940	30.0	3.0	.8382	.1859	33.0	3.0	.8273	.1758
24.0	3.5	.8942	.2387	27.0	3.5	.8783	.2273	30.0	3.5	.8690	.2167	33.0	3.5	.8570	.2055
24.0	4.0	.9275	.272	27.0	4.0	.9122	.2612	30.0	4.0	.9014	.2491	33.0	4.0	.8859	.2344
24.0	4.25	.9458	.2903	27.0	4.25	.9288	.2778	30.0	4.25	.9155	.2632	33.0	4.25	.9032	.2517
36.0	0.0	.6502	ref.	39.0	0.0	.6498	ref.	42.0	0.0	.6508	ref.				
36.0	1.0	.7020	.0518	39.0	1.0	.7013	.0515	42.0	1.0	.6990	.0482				
36.0	1.5	.7322	.0820	39.0	1.5	.7280	.0782	42.0	1.5	.7260	.0752				
36.0	2.0	.7600	.1098	39.0	2.0	.7568	.1070	42.0	2.0	.7523	.1015				
36.0	2.5	.7902	.1400	39.0	2.5	.7849	.1351	42.0	2.5	.7785	.1277				
36.0	3.0	.8188	.1686	39.0	3.0	.8119	.1621	42.0	3.0	.8036	.1528				
36.0	3.5	.8454	.1952	39.0	3.5	.8393	.1895	42.0	3.5	.8305	.1797				
36.0	4.0	.8742	.2240	39.0	4.0	.8658	.2160	42.0	4.0	.8575	.2067				
35.0	4.25	.8912	.2410	39.0	4.25	.8800	.2302	42.0	4.25	.8700	.2192				

Table II (cont.)

2ND MODE BUCKLING DATA

T	Q	M.R.	δ	T	Q	M.R.	δ	T	Q	M.R.	δ	T	Q	M.R.	δ
0.0	0.0	.4317	ref.	6.0	0.0	.4295	ref.	11.0	0.0	.4266	ref.	16.0	0.0	.4215	ref.
0.0	1.0	.4475	.0158	6.0	1.0	.4445	.0150	11.0	1.0	.4405	.0139	16.0	1.0	.4393	.0178
0.0	1.5	.4562	.0245	6.0	1.5	.4523	.0228	11.0	1.5	.4475	.0209	16.0	1.5	.4465	.0215
0.0	2.0	.4630	.0313	6.0	2.0	.4592	.0297	11.0	2.0	.4544	.0278	16.0	2.0	.4531	.0316
0.0	2.5	.4737	.0420	6.0	2.5	.4669	.0374	11.0	2.5	.4615	.0349	16.0	2.5	.4595	.0380
0.0	3.0	.4788	.0471	6.0	3.0	.4722	.0427	11.0	3.0	.4669	.0403	16.0	3.0	.4649	.0434
0.0	3.5	.4882	.0565	6.0	3.5	.4800	.0505	11.0	3.5	.4741	.0475	16.0	3.5	.4723	.0508
0.0	4.0	.4948	.0631	6.0	4.0	.4864	.0569	11.0	4.0	.4812	.0546	16.0	4.0	.4780	.0565
0.0	4.5	.5018	.0701	6.0	4.5	.4925	.0630	11.0	4.5	.4887	.0621	16.0	4.5	.4844	.0629
0.0	5.0	.5090	.0773	6.0	5.0	.4988	.0693	11.0	5.0	.4946	.0680	16.0	5.0	.4911	.0696
0.0	5.5	.5173	.0856	6.0	5.5	.5073	.0778	11.0	5.5	.5011	.0745	16.0	5.5	.4982	.0767
0.0	6.0	.5240	.0923	6.0	6.0	.5134	.0839	11.0	6.0	.5082	.0816	16.0	6.0	.5047	.0832
0.0	6.5	.5318	.1001	6.0	6.5	.5199	.0904	11.0	6.5	.5150	.0884	16.0	6.5	.5110	.0895
21.0	0.0	.4260	ref.	26.0	0.0	.4165	ref.	31.0	0.0	.4128	ref.	36.0	0.0	.4169	ref.
21.0	1.0	.4355	.0095	26.0	1.0	.4302	.0136	31.0	1.0	.4256	.0128	36.0	1.0	.4275	.0106
21.0	1.5	.4417	.0157	26.0	1.5	.4360	.0194	31.0	1.5	.4329	.0201	36.0	1.5	.4326	.0157
21.0	2.0	.4478	.0218	26.0	2.0	.4425	.0259	31.0	2.0	.4391	.0263	36.0	2.0	.4373	.0204
21.0	2.5	.4544	.0284	26.0	2.5	.4489	.0323	31.0	2.5	.4452	.0324	36.0	2.5	.4432	.0263
21.0	3.0	.4598	.0338	26.0	3.0	.4549	.0383	31.0	3.0	.4503	.0375	36.0	3.0	.4487	.0318
21.0	3.5	.4662	.0402	26.0	3.5	.4615	.0449	31.0	3.5	.4565	.0437	36.0	3.5	.4542	.0373
21.0	4.0	.4723	.0463	26.0	4.0	.4678	.0512	31.0	4.0	.4619	.0491	36.0	4.0	.4597	.0428
21.0	4.5	.4788	.0528	26.0	4.5	.4742	.0576	31.0	4.5	.4679	.0551	36.0	4.5	.4661	.0492
21.0	5.0	.4845	.0585	26.0	5.0	.4806	.0640	31.0	5.0	.4740	.0612	36.0	5.0	.4714	.0545
21.0	5.5	.4916	.0656	26.0	5.5	.4868	.0702	31.0	5.5	.4801	.0673	36.0	5.5	.4776	.0607
21.0	6.0	.4981	.0721	26.0	6.0	.4932	.0766	31.0	6.0	.4858	.0730	36.0	6.0	.4833	.0664
21.0	6.5	.5040	.0780	26.0	6.5	.4997	.0831	31.0	6.5	.4921	.0793	36.0	6.5	.4895	.0726

Note: T = Applied axial load (lbs.); Q = Applied lateral load (lbs.); M.R. = Micrometer reading (in.);
δ = Measured deflection (in.)

Table III

T	Q	M.R.	δ
41.0	0.0	.4120	ref.
41.0	1.0	.4235	.0115
41.0	1.5	.4305	.0185
41.0	2.0	.4359	.0239
41.0	2.5	.4420	.0300
41.0	3.0	.4475	.0355
41.0	3.5	.4534	.0414
41.0	4.0	.4591	.0471
41.0	4.5	.4642	.0522
41.0	5.0	.4695	.0575
41.0	5.5	.4751	.0631
41.0	6.0	.4804	.0684
41.0	6.5	.4857	.0737

T	Q	M.R.	δ
16.0	0.0	.4198	ref.
16.0	1.0	.4352	.0154
16.0	1.5	.4445	.0247
16.0	2.0	.4510	.0312
16.0	2.5	.4578	.0380
16.0	3.0	.4637	.0439
16.0	3.5	.4708	.0510
16.0	4.0	.4772	.0574
16.0	4.5	.4842	.0644
16.0	5.0	.4903	.0705
16.0	5.5	.4970	.0772
16.0	6.0	.5038	.0840
16.0	6.5	.5096	.0898

This run performed as a check
on the earlier run for T = 16.0

Table III (cont.)

DATA FOR SOUTHWELL'S METHOD

CSR	P	V	ΔV	ΔV/P
1226	16.3	- 56	49	3.00
1211	17.3	- 64	57	3.29
1197	18.3	- 72	65	3.55
1182	19.3	- 84	77	3.99
1168	20.3	- 94	87	4.29
1153	21.3	-105	98	4.60
1139	22.3	-126	119	5.33
1124	23.3	-144	137	5.88
1110	24.3	-163	156	6.42
1095	25.3	-186	179	7.08
1081	26.3	-224	217	8.25

V_o = -7mv

CSR	P	V	ΔV	ΔV/P
1226	16.3	- 61	53	3.25
1226	16.3	- 66	58	3.35
1197	18.3	- 73	65	3.55
1182	19.3	- 81	73	3.78
1168	20.3	- 92	84	4.14
1153	21.3	-106	98	4.60
1139	22.3	-123	115	5.16
1124	23.3	-140	132	5.67
1110	24.3	-162	154	6.34
1095	25.3	-197	191	7.55
1081	26.3	-227	219	8.33

V_o = -8mv

CSR	P	V	ΔV	ΔV/P
1226	16.3	- 46	45	2.76
1211	17.3	- 54	55	3.12
1197	18.3	- 65	64	3.50
1182	19.3	- 74	73	3.78
1168	20.3	- 96	95	4.68
1153	21.3	-106	105	4.93
1139	22.3	-116	115	5.16
1124	23.3	-143	142	6.09
1110	24.3	-159	158	6.50
1095	25.3	-180	179	7.08
1081	26.3	-209	208	7.91

V_o = -1mv

CSR	P	V	ΔV	ΔV/P
1226	16.3	- 55	50	3.07
1211	17.3	- 61	56	3.24
1197	18.3	- 69	64	3.50
1182	19.3	- 80	75	3.89
1168	20.3	- 89	84	4.14
1153	21.3	-106	101	4.74
1138	22.3	-121	116	5.20
1124	23.3	-138	133	5.71
1110	24.3	-160	155	6.38
1095	25.3	-181	176	6.96
1081	26.3	-211	206	7.83

V_o = -5mv

Note: CSR = Cantilever strain reading (μin/in)
P = Compressive axial load (lbs)
V = Voltage reading from digital voltmeter (mv)

An empirical equation based on earlier calibration tests was used to relate CSR and P

$$P(\text{lbs}) = (CSR_o - CSR) \times 1.0\text{lb}/14.76\mu\text{in/in} + Wt_c$$

For all tests CSR_o = 1440μin/in.
Wt_c = weight of the end cylinder = 1.5 lbs.

$\Delta V = V - V_o$

Table IV

DATA FOR TENSILE TEST TO DETERMINE YOUNG'S MODULUS

T	SRIL	SRDL	T	SIL	SDL	ASPG	STRESS
0	1920	1922	0	0	0	0	0
100	1700	1700	100	220	222	110.5	3226
200	1478	1482	200	442	440	220.5	6452
300	1255	1262	300	665	660	331.3	9677
400	1038	1043	400	882	879	440.3	12903
500	819	827	500	1101	1095	549.0	16129
600	602	610	600	1318	1312	657.5	19355
700	382	385	700	1538	1537	768.8	22581
800	161	163	800	1759	1759	879.5	25806
850	53	53	850	1867	1869	934.0	27419

Note: Strain readings (μin/in) were taken with increasing load (SRIL) and decreasing load (SRDL).

T = Tensile axial load (lbs)

ASPG = Average strain per gage

The cross-sectional area of the specimen was 0.031 in^2.

SIL = Actual strain, increasing load

SDL = Actual strain, decreasing load

Table V

B. DISCUSSION

In order to establish a point of comparison between experimental and theoretical results, the Euler load may be calculated, based on the following measured quantities:

E = Young's Modulus = $(29.4 \pm .1) \times 10^6$ psi

w = column width = $0.248 \pm .001$ in.

h = column depth = $0.125 \pm .001$ in.

L = effective column length = $19.00 \pm .01$ in.

$$P_1 = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 E}{L^2}\,\frac{wh^3}{12} = 32.44 \text{ lbs.} \tag{4.1}$$

Due to the uncertainties in the measured quantities, there is an uncertainty in the calculated value of P_1. The magnitude of this uncertainty may be approximated using a "logarithmic" error analysis. Taking the logarithmic derivative of equation (4.1)

$$\left|\frac{dP_1}{P_1}\right| = \left|\frac{dE}{E}\right| + \left|\frac{dw}{w}\right| + 3\left|\frac{dh}{h}\right| + 2\left|\frac{dL}{L}\right| \tag{4.2}$$

$$dP_1 = P_1(.0324) = 1.05 \text{ lbs.}$$

$$P_1 = 32.44 \pm 1.05 \text{ lbs.} \tag{4.3}$$

The experimental values of P_1 as determined by methods I and II fall only slightly outside the range of uncertainty associated with the calculated Euler load. The value of P_1 determined by Southwell's method may be considered as an upper bound in view of Southwell's theory [5] which indicates experimental values will be either exact or high.

The Euler load for second mode buckling is four times that for first mode. Thus, we would expect

$$P_2 = 129.32 \pm 4.20 \text{ lbs.}$$

The experimental value determined by method I is well within the range of uncertainty, but that determined by method II seems considerably in error. This discrepancy is attributable to inconsistencies in measured deflection. Referring to Table III, particularly the data for T = 11.0 and T = 16.0 lbs., it may be seen that increasing T resulted in increased deflection. For every given value of lateral load Q, the deflections measured with T = 16.0 lbs. are approximately .003 inches greater than those measured for T = 11.0 lbs. A second test with T = 16.0 lbs. was run on another day as a check, and the values obtained earlier were repeated. This discrepancy was completely unexpected in view of the theory, so several checks of the experimental apparatus were made, but no unexpected sources of error came to light. This inconsistency occurred for several (but not all) values of increased T.

Despite these anomalies, the data can be treated using method I. For a given value of axial load (T), a plot of deflection (δ) vs. lateral load (Q) is made but only the slope of this line is considered. For this reason a constant error in measured deflection has no effect on the outcome of the analysis. Method II, on the other hand, relies on the absolute rather than the relative magnitudes of measured deflections, and gives erroneous values for P_2, as expected.

The theory developed earlier, particularly equations (2.13) and (2.14), shows why it is advisable to retain several series terms when the axial loading is tensile. For any given tensile loading, the effect of additional series terms in equation (2.14) is greater than that for compressive loading (Eq. 2.13). Thus a single term approximation may be more easily justified for compressive loads than for tensile.

V. CONCLUSIONS AND RECOMMENDATIONS

Experimental determination of the primary buckling load of a bar can be accomplished without reliance on some accidental, initial imperfection. Lateral loading is used to assure dominance of the primary mode in the deflected shape. The axial load may be either tensile or compressive. By correlating observed deflection and applied lateral load for several distinct values of axial load, the buckling load may be easily and accurately determined.

If a tensile axial load is used, experimental determination of the second buckling load is possible, using appropriate lateral loads to assure dominance of the second mode in the deflected shape.

The developed theory and experimental technique are generally applicable to bars of any cross-section, although orthogonal loading schemes for determination of higher mode buckling loads may be difficult experimentally for non-prismatic bars. The use of compressive axial loading is advantageous for non-prismatic bars since it minimizes the effects of possibly unknown higher modes in the deflected shape.

It appears that techniques suggested may also be extended for use in determining plate buckling loads.

APPENDIX A - COMPUTER PROGRAM LISTINGS

The computer programs listed on the following pages were used to analyze experimental data. All programs are written in FORTRAN IV for use on an IBM 360 computer.

The programs for Method I and Southwell's Method are essentially the same. The program was taken from McCalla [3], in part, with a few additions and minor changes made as necessary.

The input required for each program, excluding data, is minimal and is explained in the body of each program. (Note that in the programs the symbol P is used to denote axial tension.)

```
      IMPLICIT REAL*8 (A-H,O-Z)
C                             ********************
C                             PROGRAM FOR METHOD I
C                             ********************
C
C
C*********************************************************************
C    THIS PROGRAM CONSISTS OF TWO PARTS.  THE FIRST IS DESIGNED TO PLOT
C    DELTA,THE OBSERVED DEFLECTION, VS. Q, THE LATERAL LOAD, FOR EACH
C    VALUE OF P, THE AXIAL LOAD. THIS IS DONE USING A LINEAR LEAST SQUARES
C    FIT.THE INVERSE SLOPE OF EACH OF THESE LINES IS THE VALUE OF K'.
C    THE SECOND PART OF THIS PROGRAM IS DESIGNED TO PLOT K' VS. P, AGAIN
C    USING A LINEAR LEAST SQUARES FIT.  IT THEN PRINTS OUT A NEW VALUE OF
C    K',TAKEN FROM THE FITTED LINE.  IT ALSO PRINTS OUT THE BEST VALUE OF
C    P CRITICAL(INTERCEPT DIVIDED BY SLOPE).
C*********************************************************************
C
C    N=NUMBER OF DATA POINTS MINUS ONE,LL IS THE ORDER OF THE POLYNOMIAL
C    DESIRED, IN THIS CASE 1.
C
      DIMENSION X(12),Y(12),XC(2),YX(2),A(2),B(2),C(2,2),L(2),M(2)
      DIMENSION R(15),S(15)
      MM=0
      READ(5,99) MODENO,NDP
    5 READ(5,100) N,LL,P
      MM=MM+1
      IF(N.EQ.0) GO TO 60
      NP1=N+1
      MP1=LL+1
      M2=LL*2
      READ(5,101) (X(I),Y(I),I=1,NP1)
C
C    FORM SUMS OF POWERS OF X(I)
C
      DO 20 K=1,M2
      XC(K)=0.0
      DO 15 I=1,NP1
   15 XC(K)=XC(K)+X(I)**K
   20 CONTINUE
C
C    FORM SUM OF Y(I)
C
      YC=0.0
      DO 22 I=1,NP1
   22 YC=YC+Y(I)
C
C    FORM SUMS OF PRODUCTS Y(I)*X(I)**K
C
```

```
    DO 30 K=1,LL
    YX(K)=0.0
    DO 25 I=1,NP1
  25 YX(K)=YX(K)+Y(I)*X(I)**K
  30 CONTINUE
C
C  GENERATE NORMAL MATRIX C USING SUMS OF POWERS OF X(I)
C
    DO 40 I=1,MP1
    DO 35 J=1,MP1
    IPJM2=I+J-2
    IF(IPJM2) 33,31,33
  31 C(1,1)=DFLOAT(NP1)
    GO TO 35
  33 C(I,J)=XC(IPJM2)
  35 CONTINUE
  40 CONTINUE
C
C  GENERATE RIGHT-SIDE MATRIX B
C
    B(1)=YC
    DO 45 I=2,MP1
  45 B(I)=YX(I-1)
C
C  INVERT NORMAL MATRIX C
C
    CALL DMINV(C,2,DET,L,M)
C
C  C IS NOW THE INVERSE OF C
C  FORM MATRIX PRODUCT OF INVERSE AND RIGHT-SIDE MATRIX
C
    DO 55 I=1,MP1
    A(I)=0.0
    DO 54 J=1,MP1
  54 A(I)=A(I)+C(I,J)*B(J)
  55 CONTINUE
    CAYP=1.0/A(2)
    R(MM)=P
    S(MM)=CAYP
    WRITE(6,112) P,CAYP
    WRITE(6,111)
    WRITE(6,110) (A(K),K=1,MP1)
    GO TO 5
C
C  THE SECOND PART OF THE PROGRAM STARTS HERE
C  THE INPUT IS TAKEN FROM THE FIRST PART
C  OF THE PROGRAM. THIS SECTION FINDS THE BEST
C  FITTED LINE FOR K PRIME VS. AXIAL LOAD.
```

```
C
   60 READ(5,200) N,LL
      NP1=N+1
      MP1=LL+1
      M2=LL*2
C
C     FORM SUMS OF POWERS OF R(I)
C
      DO 70 K=1,M2
      XC(K)=0.0
      DO 65 I=1,NP1
   65 XC(K)=XC(K)+R(I)**K
   70 CONTINUE
C
C     FORM SUM OF S(I)
C
      YC=0.0
      DO 72 I=1,NP1
   72 YC=YC+S(I)
C
C     FORM SUMS OF PRODUCTS S(I)*R(I)**K
C
      DO 80 K=1,LL
      YX(K)=0.0
      DO 75 I=1,NP1
   75 YX(K)=YX(K)+S(I)*R(I)**K
   80 CONTINUE
C
C     GENERATE NORMAL MATRIX C USING SUMS OF POWERS OF R(I)
C
      DO 85 I=1,MP1
      DO 83 J=1,MP1
      IPJM2=I+J-2
      IF(IPJM2)  82,81,82
   81 C(1,1)=DFLOAT(NP1)
      GO TO 83
   82 C(I,J)=XC(IPJM2)
   83 CONTINUE
   85 CONTINUE
C
C     GENERATE RIGHT-SIDE MATRIX B
C
      B(1)=YC
      DO 87 I=2,MP1
   87 B(I)=YX(I-1)
C
C     INVERT NORMAL MATRIX C
C
```

```
      CALL DMINV(C,2,DET,L,M)
C
C  C IS NOW THE INVERSE OF C
C  FORM MATRIX PRODUCT OF INVERSE AND RIGHT-SIDE MATRIX
C
      DO 90 I=1,MP1
      A(I)=0.0
      DO 89 J=1,MP1
   89 A(I)=A(I)+C(I,J)*B(J)
   90 CONTINUE
      WRITE(6,214)
      WRITE(6,211)
      WRITE(6,210)(A(K),K=1,MP1)
      DO 95 I=1,NP1
      S(I)=A(2)*R(I)+A(1)
   95 WRITE(6,212) R(I),S(I)
      PC=A(1)/A(2)
      WRITE(6,216) MODENO,NDP
      WRITE(6,213) PC
      WRITE(6,215)  A(1)
   99 FORMAT(I2,I3)
  100 FORMAT(I3,I2,F10.6)
  101 FORMAT(2F14.6)
  110 FORMAT(1X,E14.6)
  111 FORMAT(18H COEFFICIENTS A(K))
  112 FORMAT(/,' AXIAL LOAD=',F10.6,'    K PRIME=',F10.6)
  200 FORMAT(I3,I2)
  201 FORMAT(2F14.6)
  210 FORMAT(1X,E14.6)
  211 FORMAT(18H COEFFICIENTS A(K))
  212 FORMAT(/,' AXIAL LOAD=',F10.6,'    K PRIME=',F10.6)
  213 FORMAT(/,' THE AVE. VALUE OF PC IS',F13.6)
  214 FORMAT(/,' *******************************************************
     1*******************',/)
  215 FORMAT(/,' THE BEST VALUE OF K IS',F6.3)
  216 FORMAT(/,' DATA IS FOR MODE',I2,' BUCKLING.  THE NO. OF DATA PTS.
     1IS',I4)
      END
```

```
      IMPLICIT REAL*8(A-H,O-Z)
C                             *********************
C                             PROGRAM FOR METHOD II
C                             *********************
C
C
C***********************************************************************
C   THIS PROGRAM IS DESIGNED TO FIND THE BEST VALUE OF P CRITICAL FOR ANY
C   MODE BY MINIMIZING THE SUM OF THE SQUARES.  THE EXPERIMENTAL VALUES
C   OF LATERAL LOAD (Q), AXIAL LOAD (P), AND DEFLECTION (DEL) ARE READ
C   IN AS DATA FOR THE PROGRAM.     SIXTY VALUES OF THE PARAMETER PCR
C   (P CRITICAL) ARE USED.  THE BEST VALUE OF PCR YIELDS A MINIMUM VALUE
C   OF THE SUM OF THE SQUARES.
C***********************************************************************
      DIMENSION Q(200),P(200),DEL(200),E(200),PCR(60),S(60),RHO(60),
     1C(200)
C
C   MODENO= NO. OF THE MODE UNDER CONSIDERATION, NDP=NO. OF DATA POINTS.
C
      READ(5,98) MODENO
      READ(5,99) NDP
      READ(5,100)(Q(I),P(I),DEL(I),I=1,NDP)
      READ(5,101) (PCR(I),I=1,60)
      WRITE(6,201) MODENO,NDP
      WRITE(6,202)
      DO 80 N=1,60
C
C   INITIALISE A AND B
C
      A1=0.0D0
      B1=0.0D0
C
C   THE VALUE OF C MAY BE DETERMINED USING ANY NO. OF SERIES TERMS.
C
      DO 10 I=1,NDP
   10 C(I)=(1.0D0/(1.0D0+(P(I)/PCR(N))))+(1.0D0/(81.0D0+(9.0D0*P(I)/PCR(
     1N))))+(1.0D0/(625.0D0+(25.0D0*P(I)/PCR(N))))+(1.0D0/(2401.0D0+(
     249.0D0*P(I)/PCR(N))))
C
C   PERFORM THE NECESSARY SUMMATIONS
C
      DO 20 I=1,NDP
      A1=A1+Q(I)*C(I)*DEL(I)
   20 B1=B1+(Q(I)**2)*(C(I)**2)
C
C   FIND RHO (1/K)
C
      RHO(N)=A1/B1
```

```
      SUMSQ=0.0D0
C
C   FIND THE ERROR ASSOCIATED WITH THE I TH DATA POINT
C   SUMSQ IS THE SUM OF THE SQUARE  OF THE ERRORS.
C
      DO 30 I=1,NDP
      E(I)=Q(I)*RHO(N)*C(I)-DEL(I)
   30 SUMSQ=SUMSQ+E(I)**2
      S(N)=SUMSQ
      WRITE(6,200) N,PCR(N),RHO(N),S(N)
C
C   THE LOOP ENDING WITH STATEMENT 80 IS DONE FOR SIXTY VALUES OF PCR
C
   80 CONTINUE
      SMIN=1000.0
C
C   AFTER EXECUTION FOR ALL VALUES OF PCR, FIND THE MINIMUM VALUE OF THE
C   SUM OF THE SQUARES, AND DIRECTLY THE BEST VALUE OF P CRITICAL
C
      DO 85 I=1,60
      IF(S(I).LT.SMIN) BESTR=RHO(I)
      IF(S(I).LT.SMIN) BESTP=PCR(I)
      IF(S(I).LT.SMIN) SMIN=S(I)
   85 CONTINUE
      BESTR=1.0D0/BESTR
      WRITE(6,203) BESTR
      WRITE(6,205) BESTP
      WRITE(6,206) SMIN
      WRITE(6,207)
      WRITE(6,208)
      WRITE(6,209)(Q(I),P(I),DEL(I),I=1,NDP)
   98 FORMAT(I2)
   99 FORMAT(I5)
  100 FORMAT(3F10.6)
  101 FORMAT(F10.6)
  200 FORMAT(/,I5,3F15.6)
  201 FORMAT(' DATA IS FOR MODE',I2,' BUCKLING.  NO. OF DATA PTS IS:',
     1I5,/)
  202 FORMAT(/,'  RUN',12X,'PCR', 12X,'RHO',5X,'SUM OF SQS',/)
  203 FORMAT(/,' BEST VALUE OF K IS:',F15.6)
  205 FORMAT(/,' BEST VALUE OF PCR IS:',F15.6)
  206 FORMAT(/,' MIN. VALUE OF SUM OF THE SQS. IS:',F15.6)
  207 FORMAT(/,'*************************************  DATA  ***********
     1*******************',/)
  208 FORMAT(/,10X,'Q',15X,'P',13X,'DEL',/)
  209 FORMAT(3F15.6)
      END
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
C                    ******************************
C                    PROGRAM FOR SOUTHWELL'S METHOD
C                    ******************************
C
C
C************************************************************************
C   THIS PROGRAM IS DESIGNED TO FIND THE BEST FITTING STRAIGHT LINE FOR
C   DELTA V/LOAD VS. DELTA V  (SOUTHWELL PLOT). THIS IS DONE USING A
C   LINEAR LEAST SQUARES FIT.  THE INVERSE SLOPE OF THIS LINE IS THE
C   FIRST CRITICAL BUCKLING LOAD.  THIS IS DONE FOR EACH OF FOUR RUNS,
C   AND THESE FOUR VALUES ARE THEN AVERAGED.
C************************************************************************
C
C   N=NUMBER OF DATA POINTS MINUS ONE,LL IS THE ORDER OF THE POLYNOMIAL
C   DESIRED.
C
      DIMENSION X(25),Y(25),XC(2),YX(2),A(2),B(2),C(2,2),L(2),M(2)
      D=0.0D0
      SUM=0.0D0
    5 READ(5,100) N,LL
      IF(N.EQ.0) GO TO 60
      D=D+1.0D0
      MM=D
      NP1=N+1
      MP1=LL+1
      M2=LL*2
      READ(5,101)(X(I),Y(I),I=1,NP1)
C
C   FORM SUMS OF POWERS OF X(I)
C
      DO 20 K=1,M2
      XC(K)=0.0D0
      DO 15 I=1,NP1
   15 XC(K)=XC(K)+X(I)**K
   20 CONTINUE
C
C   FORM SUM OF Y(I)
C
      YC=0.0D0
      DO 22 I=1,NP1
   22 YC=YC+Y(I)
C
C   FORM SUMS OF PRODUCTS Y(I)*X(I)**K
C
      DO 30 K=1,LL
      YX(K)=0.0D0
      DO 25 I=1,NP1
```

```
   25 YX(K)=YX(K)+Y(I)*X(I)**K
   30 CONTINUE
C
C  GENERATE NORMAL MATRIX C USING SUMS OF POWERS OF X(I)
C
      DO 40 I=1,MP1
      DO 35 J=1,MP1
      IPJM2=I+J-2
      IF(IPJM2) 33,31,33
   31 C(1,1)=DFLOAT(NP1)
      GO TO 35
   33 C(I,J)=XC(IPJM2)
   35 CONTINUE
   40 CONTINUE
C
C  GENERATE RIGHT-SIDE MATRIX B
C
      B(1)=YC
      DO 45 I=2,MP1
   45 B(I)=YX(I-1)
C
C  INVERT NORMAL MATRIX C
C
      CALL DMINV(C,2,DET,L,M)
C
C  C IS NOW THE INVERSE OF C
C  FORM MATRIX PRODUCT OF INVERSE AND RIGHT-SIDE MATRIX
C
      DO 55 I=1,MP1
      A(I)=0.0D0
      DO 54 J=1,MP1
   54 A(I)=A(I)+C(I,J)*B(J)
   55 CONTINUE
      PCRIT=1.0D0/A(2)
      WRITE(6,201) MM,PCRIT
      WRITE(6,202)
      WRITE(6,203)
      WRITE(6,204)(X(I),Y(I),I=1,NP1)
      SUM=SUM+PCRIT
      GO TO 5
   60 PCRITA=SUM/D
      WRITE(6,205) MM,PCRITA
  100 FORMAT(I3,I2)
  101 FORMAT(2F14.6)
  201 FORMAT(/,' THE VALUE OF THE FIRST CRITICAL LOAD FOR RUN',I2,' IS',
     1F11.6)
  202 FORMAT(/,' *************************** DATA ********************',/)
  203 FORMAT(/,'      DELTA V     DELTA V/LOAD',/)
```

```
204 FORMAT(/,2F14.6)
205 FORMAT(/,' THE AVERAGE VALUE OF P CRITICAL FOR',I2,' RUNS IS: PCR=
   1',F7.3)
    END
```

APPENDIX B - EVALUATION OF INTEGRALS INVOLVING ORTHOGONAL FUNCTIONS AND THEIR DERIVATIVES

The equation and conditions governing the shape of the strained centerline of a pin-ended column subjected to an axial compressive load P are

$$BY'' + PY = 0 \text{ where } B = B(x) = EI(x) \qquad \text{(B-1a)}$$

$$Y(0) = Y(L) = 0 \qquad \text{(B-1b)}$$

Let (Y_i, P_i) and (Y_j, P_j) be two distinct solutions to this equation

$$BY_i'' + P_iY_i = 0$$

$$BY_j'' + P_jY_j = 0 \qquad \text{(B-2)}$$

Multiplying both sides of equation B-2 by Y_j''

$$\int_0^L BY_i''\, Y_j''\, dx + \int_0^L P_iY_iY_j''\, dx = 0$$

$$-P_i\int_0^L Y_iY_j''\, dx = \int_0^L BY_i''Y_j''\, dx = -P_j\int_0^L Y_i''Y_j\, dx \qquad \text{(B-3)}$$

$$\therefore P_i\int_0^L Y_iY_j''\, dx = P_j\int_0^L Y_i''Y_j\, dx \qquad \text{(B-4)}$$

$$= P_j\left\{\left[Y_jY_i\right]_0^L - \int_0^L Y_i'Y_j'\, dx\right\}$$

$$= -P_j\int_0^L Y_i'Y_j'\, dx$$

Also the left side of equation B-4 is

$$P_i\int_0^L Y_iY_j''\, dx = -P_i\int_0^L Y_i'Y_j'\, dx$$

$$\therefore (P_i - P_j)\int_0^L Y_i'Y_j'\, dx = 0$$

But P_i and P_j are distinct ($i \neq j$)

$$\therefore \int_0^L Y_i' Y_j'' \, dx = 0 \qquad i \neq j \tag{B-5}$$

Also, from equation B-4

$$\int_0^L Y_i'' Y_j \, dx = -\int_0^L Y_i'' Y_j \, dx = 0 \qquad i \neq j \tag{B-6}$$

By equation B-3

$$\int_0^L B Y_i'' Y_j'' \, dx = -P_j \int_0^L Y_i'' Y_j \, dx = 0 \qquad i \neq j \tag{B-7}$$

Summarizing the preceeding arguments

$$\int_0^L Y_i Y_j'' \, dx = I_i \delta_{ij}; \; I_i = \int_0^L Y_i Y_i'' \, dx$$

$$\int_0^L Y_i' Y_j' \, dx = J_i \delta_{ij}; \; J_i = \int_0^L (Y_i')^2 dx$$

$$\int_0^L B Y_i'' Y_j'' dx = K_i \delta_{ij}; \; K_i = \int_0^L (Y_i'')^2 dx$$

where δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Returning to equation B-3 and B-4 it may be seen that

$$K_i = P_i J_i$$

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13. ABSTRACT

If an elastic bar with some initial imperfection is subjected to increasing compressive axial load, the lateral deflection, measured anywhere along its length, increases monotonically; the primary buckling mode becoming more and more dominant in the deflected shape. A classical approximate technique, due to R. V. Southwell, correlates axial load and lateral deflection for determination of the primary buckling load of the bar. However, Southwell's approximation may be inaccurate if the primary mode does not predominate in the deflected shape. The present thesis proposes a technique which employs axial loading, but also transverse loading to insure dominance of the primary mode in the deflected shape.

The technique allows experimental determination of the second buckling load, again using appropriate transverse loads to insure dominance of that mode in the deflected shape.

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14 KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
column buckling						
column experiments						
Southwell's column method						

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